JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 41, 411–419 (1973)

A Generalization of some Boundedness Results by Reissig and Tejumola

J. O. C. EZEILO

Department of Mathematics, University of Nigeria, Nsukka

Submitted by Richard Bellman

1. We shall be concerned here with the differential equation

$$x''' + \{f(x') + g(x, x')\} x'' + \varphi_2(x') + \varphi_3(x) = \psi(t, x, x', x''), \tag{1.1}$$

in which f, g, φ_2, φ_3 and ψ are continuous functions depending only on the arguments shown and $(\partial g/\partial x)(x, y)$ and $\varphi_3'(x)$ exist and are continuous for all values of x and y.

The special case

$$x''' + g(x, x') x'' + \varphi_2(x') + \varphi_3(x) = \psi(t)$$

(corresponding to $f \equiv 0$ in (1.1)) in which ψ is a bounded function depending only on t has been examined by Reissig [1]. His results there show that all solutions of this particular equation are ultimately bounded if

$$\varphi_3(x) \operatorname{sgn} x > 0 \quad (x \neq 0) \qquad \text{and} \qquad \varphi_3(x) \operatorname{sgn} x \to +\infty \quad \text{as} \quad |x| \to \infty,$$

and if further, there are constants a', a, b, c, α such that

$$\varphi_3'(x) \leqslant c \qquad \text{for all } x,$$

$$\frac{\varphi_2(y)}{y} \geqslant b > 0 \quad (y \neq 0), \qquad g(x, y) \geqslant a > \frac{c}{b} > 0,$$

$$y \frac{\partial g}{\partial x}(x, y) \leqslant 0 \qquad \text{and} \qquad g(x, y)\, e^{-\alpha|y|} \leqslant a' \quad (a' \geqslant a, \alpha \geqslant 0).$$

Tejumola, in a private communication, has informed me of his treatment involving another special case

$$x''' + f(x') x'' + \varphi_2(x') + \varphi_3(x) = \psi(t, x, x', x'')$$

(corresponding to $g \equiv 0$ in (1.1)) in which ψ satisfies

$$|\psi(t, x, y, z)| \leqslant A + \epsilon(|y| + |z|), \qquad \text{for all } t, x, y \text{ and } z, \tag{1.2}$$

with $A \geqslant 0$ and with $\epsilon \geqslant 0$ sufficiently small. His result, the proof of which is to appear in the paper [2], shows that the ultimate boundedness property of solutions holds too for this special equation if f, φ_2 and φ_3 satisfy the conditions:

$$f(y) \geqslant \delta_1 \qquad \text{for all } y; \qquad \varphi_2(y)/y \geqslant \delta_2 \qquad (|y| \geqslant \Delta_2 > 0)$$
$$\varphi_3'(x) \leqslant \delta_3 < \delta_1\delta_2 \qquad (|x| \geqslant \Delta_3 > 0)$$

and

$$\varphi_3(x) \operatorname{sgn} x \to +\infty \qquad \text{as} \qquad |x| \to \infty,$$

where δ_1, δ_2 and δ_3 are positive constants. It should be pointed out here that, inspite of the more general nature of the coefficient function g in Reissig's equation (given earlier on), his result does not really generalize or supersede that of Tejumola. Indeed, when Reissig's results are specialized to the case g independent of x (as in Tejumola's), the boundedness property of solutions holds only if g further satisfies

$$g(y) \leqslant a' e^{\alpha|y|},$$

and this condition does not at all feature in Tejumola's treatment.

My object in considering (1.1) in the present paper is to show that Reissig's and Tejumola's results are each a special case of the following single theorem:

THEOREM. *Suppose that*

(i) *there is a constant* $\delta_1 > 0$ *such that*

$$f(y) + g(x, y) \geqslant \delta_1 \qquad \textit{for all } x \textit{ and } y,$$

(ii) *there are constants* $\delta_2 > 0$, $\Delta_2 > 0$ *such that*

$$\varphi_2(y)/y \geqslant \delta_2 \qquad (|y| \geqslant \Delta_2),$$

(iii) *there is a constant* $\Delta_3 > 0$ *such that* $\varphi_3'(x) \leqslant \delta_3$ *for* $|x| \geqslant \Delta_3$ *where* δ_3 *is a constant such that*

$$\delta_1\delta_2 > \delta_3 > 0, \tag{1.3}$$

(iv) $y(\partial g/\partial x)(x, y) \leqslant 0$, *and there exist constants* $\alpha_0 \geqslant 0$, $\alpha \geqslant 0$ *and* $\rho \geqslant 0$ *such that*

$$|g(x, y)| e^{-\alpha|y|} \leqslant a_0 + \rho |y|, \tag{1.4}$$

for all x and y,

(v) $\varphi_3(x) \operatorname{sgn} x \to +\infty$ *as* $|x| \to \infty$,

(vi) $\psi(t, x, y, z)$ *satisfies* (1.2).

Then there exist constants $\epsilon_0 > 0$, $D_0 > 0$ whose magnitudes depend only on A, f, g, φ_2 and φ_3 such that, if $\epsilon \leqslant \epsilon_0$ and $\rho \leqslant \epsilon_0$ then every solution $x(t)$ of (1.1) *ultimately satisfies*

$$|x(t)| \leqslant D_0, \qquad |x'(t)| \leqslant D_0, \qquad |x''(t)| \leqslant D_0. \tag{1.5}$$

Observe the slightly improved bound here (1.4) for $ge^{-\alpha|y|}$.

Also observe that, when results are specialized to the case $f \equiv 0$ and ψ depending only on t, as in [1], the present theorem makes use of conditions on $\varphi_3(x) \operatorname{sgn} x$, $\varphi_3'(x)$ and $\varphi_2(y)$ which hold only for sufficiently large x and y, whereas [1] requires these conditions to hold for all x, y.

For the special case $f \equiv$ constant and $g \equiv$ constant the present theorem reduces to an earlier boundedness theorem in [3].

2. We shall adopt the notation in [3] with regard to the various constants which will feature in the proofs here. Thus the D's in what follows are positive constants whose magnitudes depend only on A, f, g, φ_2 and φ_3, subject to the usual convention that the unnumbered D's are not the same in each place of occurrence whereas all the D's: D_1, D_2, D_3,... with suffices attached retain a fixed identity throughout. The dependence of a D on any extra argument will be denoted by displaying the extra argument explicitly: Thus, for example, $D(\epsilon)$ in any context here will stand for a positive constant whose magnitude depends on $A, f, g, \varphi_2, \varphi_3$ and ϵ.

3. It is convenient to replace (1.1) by the equivalent system

$$\begin{gathered} x' = y, \qquad y' = z, \\ z' = -\{f(y) + g(x, y)\}\, z - \varphi_2(y) - \varphi_3(x) + \psi(t, x, y, z) \end{gathered} \tag{3.1}$$

derived as a result of setting $y = x'$, $z = x''$ in (1.1).

To prove the theorem it will be sufficient, for the same reasons as in Section 3 of [3], to show that there is a continuous function $V(x, y, z)$ satisfying

$$V(x, y, z) \to +\infty \qquad \text{as} \qquad x^2 + y^2 + z^2 \to \infty \tag{3.2}$$

such that the limit

$$V'^* \equiv \limsup_{h \to +0} \frac{V(x(t+h), y(t+h), z(t+h)) - V(x(t), y(t), z(t))}{h} \tag{3.3}$$

exists, corresponding to any solution $(x(t), y(t), z(t))$ of (3.1), and satisfies

$$V'^* \leqslant -D_1 \qquad \text{if} \qquad x^2(t) + y^2(t) + z^2(t) \geqslant D_2 \tag{3.4}$$

for some constants $D_1 > 0$, $D_2 > 0$.

4. A Function V

Following the pattern in [3] it remains now to exhibit a V with the requisite properties.

The actual V to be used here is derived from an adaptation of certain functions in [1] and [3]. Let

$$C \equiv \max_{|x| \leqslant \Delta_3} |\varphi_3'(x)|$$

and let $\delta > 0$ be a constant, fixed, as is possible in view of (1.4), such that

$$\delta_2 \delta_3^{-1} > \delta > \delta_1^{-1}. \tag{4.1}$$

Further, let $\chi_3 = \chi_3(x)$ be the differentiable function given by

$$\chi_3 = \begin{cases} \operatorname{sgn} x, & \text{if} \quad |x| \geqslant 2\Delta_3 \\ \sin \pi x/(4\Delta_3), & \text{if} \quad |x| \leqslant 2\Delta_3 \end{cases} \tag{4.2}$$

and let $\chi_2 = \chi_2(x, y, z)$ be the continuous function defined, for $|x| \geqslant 1$, by

$$\chi_2 = \begin{cases} z/L \operatorname{sgn} x, & \text{if} \quad |z| \leqslant L, \\ \operatorname{sgn} x \operatorname{sgn} z, & \text{if} \quad |z| \geqslant L, \end{cases} \tag{4.3}$$

and, for $|x| \leqslant 1$, by

$$\chi_2 = \begin{cases} zx/L, & \text{if} \quad |z| \leqslant L, \\ x \operatorname{sgn} z, & \text{if} \quad |z| \geqslant L, \end{cases} \tag{4.4}$$

where $L \geqslant 1$ is a constant whose magnitude will be fixed later to advantage.

Consider now the function $V = V(x, y, z)$ given by

$$V = V_1 + V_2 + V_3, \tag{4.5}$$

where

$$\begin{aligned} 2V_1 = 2 \int_0^x \varphi_3(\xi)\, d\xi + \delta \left(2 \int_0^y \varphi_2(\eta)\, d\eta + z^2\right) \\ + 2 \int_0^y \eta\{f(\eta) + g(x, \eta)\}\, d\eta + 2\delta y \varphi_3(x) + 2yz, \end{aligned} \tag{4.6}$$

$$V_2 = \chi_2(x, y, z + F(y))\, e^{-\alpha |y|}, \qquad \left(F(y) \equiv \int_0^y f(\eta)\, d\eta\right), \tag{4.7}$$

and

$$V_3 = D_3 y \chi_3(x), \qquad D_3 \equiv 8\Delta_3 \delta_2 C/(\pi \delta_3). \tag{4.8}$$

Observe that V_1 is an adaptation of the function U_1, and V_3 identical with the function U_3, on p. 737 of [3]. Also the function V_2 is essentially the same as the function V_2 in [1] except that we have here $z + F(y)$ in place of z.

It will now be shown that, subject to the given condition on f, g, φ_2, φ_3 and ψ, the function V given by (4.5), (4.6), (4.7) and (4.8) satisfies the conditions (3.2) and (3.4). The proof of this will be in two stages (Sections 5–7 to follow).

5. Verification of (3.2)

It is clear from (4.3), (4.4) and (4.7) that $| V_2 | \leqslant 1$ and from (4.2) and (4.8) that $| V_3 | \leqslant D_3 | y |$. Also, since $f(y) + g(x, y) \geqslant \delta_1 > 0$ for all x and y,

$$\int_0^y \eta\{f(\eta) + g(x, \eta)\}\, d\eta \geqslant \delta_1 y^2.$$

Hence, by (4.5) and (4.6),

$$2V \geqslant 2W_1 - 2D_3 | y | - D, \tag{5.1}$$

where

$$2W_1 \equiv 2 \int_0^x \varphi_3(\xi)\, d\xi + \delta \left(2 \int_0^y \varphi_2(\eta)\, d\eta + z^2\right) + \delta_1 y^2 + 2\delta y \varphi_3(x) + 2yz.$$

The function W_1 here is identical with the function U_1 given by (4.4) in Section 4 of [3] except only that we have here δ_1 in place of α. The estimate for this U_1 in Section 5 of [3] shows that here

$$2W_1 \geqslant \delta(z + \delta^{-1} y)^2 + (\delta_1 - \delta^{-1})\, y^2 + \delta_2^{-1} I_2(x) - D, \tag{5.2}$$

where $I_2(x)$ satisfies

$$I_2(x) \to +\infty \qquad \text{as} \qquad | x | \to \infty. \tag{5.3}$$

Since the coefficient $(\delta_1 - \delta^{-1})$ of y^2 in (5.2) is positive, by (4.1), the result (3.2) follows at once from (5.1), (5.2) and (5.3).

6. Verification of (3.4): Some Preliminary Calculations

It is clear from the forms of V_1, V_2 and V_3 and from the regularity conditions on f, g, φ_2 and φ_3 that V'^* exists; and a straightforward calculation from (3.1), (4.2)–(4.8) will show that its value can in fact be set out in the form

$$V'^* = U_1 + U_2, \tag{6.1}$$

where

$$
\begin{aligned}
U_1 = &- \{\delta(f+g) - 1\}\, z^2 - \{y\varphi_2 + [D_3\chi_3' - \delta\varphi_3']\, y^2\} \\
&+ y \int_0^y \eta \frac{\partial g}{\partial x}(x, \eta)\, d\eta + (y + \delta z)\, \psi + D_3 z \chi_3 ,
\end{aligned}
\tag{6.2}
$$

and

$$
U_2 = e^{-\alpha |y|} W_2 , \tag{6.3}
$$

W_2 being the function defined, for $|x| \geqslant 1$, by

$$
W_2 = \begin{cases} -\dfrac{1}{L}(zg + \varphi_2 + \varphi_3 - \psi)\operatorname{sgn} x - \dfrac{\alpha}{L} z(z+F)\operatorname{sgn} x \operatorname{sgn} y, \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if} \quad |z+F| \leqslant L \\ -\alpha z \operatorname{sgn} x \operatorname{sgn} y \operatorname{sgn}(z+F), \qquad \text{if} \quad |z+F| \geqslant L, \end{cases} \tag{6.4}
$$

and, for $|x| \leqslant 1$, by

$$
W_2 = \begin{cases} \dfrac{1}{L}(z+F)\, y - \dfrac{1}{L}(zg + \varphi_2 + \varphi_3 - \psi)\, x - \alpha x z (z+F)\operatorname{sgn} y, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if} \quad |z+F| \leqslant L, \\ y \operatorname{sgn}(z+F) - \alpha z x \operatorname{sgn} y \operatorname{sgn}(z+F), \qquad \text{if} \quad |z+F| \geqslant L. \end{cases} \tag{6.5}
$$

The component U_1 here arises from the functions V_1 and V_3, and the component U_2 from the component V_2 of V. By using

$$
L \geqslant 1, \qquad |z+F| \geqslant L, \qquad |z+F| \leqslant L, \qquad |x| \leqslant 1,
$$

as required in (6.4) and (6.5), it is easy to verify for the function W_2 that for $|x| \geqslant 1$,

$$
W_2 \leqslant \begin{cases} |z|\,|g| + |\varphi_2| + \alpha |z| + |\psi| - \dfrac{1}{L}\varphi_3(x)\operatorname{sgn} x, \\ \qquad\qquad \text{if} \quad |z+F| \leqslant L \\ \alpha |z| , \qquad \text{if} \quad |z+F| \geqslant L \end{cases}
$$

and, for $|x| \leqslant 1$,

$$
W_2 \leqslant \begin{cases} |z|\,|g| + |\varphi_2| + |\psi| + D(|y| + |z| + 1), \\ \qquad\qquad\qquad \text{if} \quad |z+F| \leqslant L, \\ |y| + \alpha |z| , \qquad \text{if} \quad |z+F| \geqslant L, \end{cases}
$$

where, in the first line of the latter inequality for W_2 we have exploited the continuity of φ_3 to majorize $|\varphi_3|$ by D for $|x| \leqslant 1$. Hence, by (1.2) and (6.3), we have that, for $|x| \geqslant 1$,

$$U_2 \leqslant \begin{cases} A + \dfrac{1}{2}\rho(y^2 + z^2) + D(\epsilon)(|y| + |z|) + |\varphi_2| - \dfrac{e^{-\alpha|y|}}{L}\varphi_3(x)\operatorname{sgn} x, \\ \qquad\qquad \text{if} \quad |z + F| \leqslant L, \\ \alpha|z|, \quad \text{if} \quad |z + F| \geqslant L, \end{cases} \tag{6.6}$$

and, for $|x| \leqslant 1$,

$$U_2 \leqslant \begin{cases} A + \frac{1}{2}\rho(y^2 + z^2) + D(\epsilon)(|y| + |z| + 1) + |\varphi_2|, \\ \qquad\qquad \text{if} \quad |z + F| \leqslant L \\ |y| + \alpha|z|, \quad \text{if} \quad |z + F| \geqslant L. \end{cases} \tag{6.7}$$

Coming now to U_1 (see (6.2)) we observe first that the integral term is nonnegative since $\eta(\partial g/\partial x)(x, \eta) \leqslant 0$, and then that

$$\{\delta(f + g) - 1\} z^2 \geqslant D_4 z^2$$

for some constant D_4, since $f + g \geqslant \delta_1$ implies here that

$$\delta(f + g) - 1 \geqslant \delta\delta_1 - 1 > 0,$$

by (4.1). Thus, since $|\chi_3(x)| \leqslant 1$, from the definition (4.2), we obtain, from (6.2), after setting

$$y\varphi_2 + \{D_3\chi_3' - \delta\varphi_3'\} y^2 \equiv W_3,$$

that

$$\begin{aligned} U_1 &\leqslant -D_4 z^2 - W_3(x, y) + D|z| + D(|y| + |z|)|\psi| \\ &\leqslant -D_4 z^2 - W_3(x, y) + D(|y| + |z|) + \epsilon D_5(y^2 + z^2), \end{aligned} \tag{6.8}$$

for some constant D_5, by (1.2).

7. Completion of the Verification of (3.4)

It remains now to examine more closely the functions arising from a combination of the function on the right hand side of (6.8) with *each* of the four functions on the righthandsides of (6.6) and (6.7), and to show that the sign in each case is negative provided that $x^2 + y^2 + z^2$ is large enough.

For this we shall need some more definite information about the two functions:

$$-W_3(x, y) \qquad \text{and} \qquad -W_3(x, y) + |\varphi_2(y)|,$$

the latter of which arises from a combination of (6.8) with the first inequality in either of (6.6), (6.7). Fresh estimates here of these two functions, however, appear quite unnecessary since the function W_3 is in fact the same as the function W which has been estimated in great detail in Section 6 of [3] in connection with the proof of Lemma 2 of [3]. The calculations there show that, with the definitions of χ_3 and D_3 as given in (4.2) and (4.8), respectively, there exists a constant D_6 such that

$$-W_3(x, y) \leqslant -D_6 y^2 + D, \tag{7.1}$$

$$-W_3(x, y) + |\varphi_2(y)| \leqslant -D_6 y^2 + D(|y| + 1), \tag{7.2}$$

for all x, y.

In what follows assume ϵ, ρ fixed such that

$$\epsilon \leqslant \tfrac{1}{4} D_5^{-1} \min(D_4, D_6), \qquad \rho \leqslant \tfrac{1}{2} \min(D_4, D_6). \tag{7.3}$$

Then, by combining (6.6)–(6.8) with (6.1) and then using (7.1) and (7.2) as required, we shall arrive at the following estimate for V'^*: For $|x| \geqslant 1$,

$$V'^* \leqslant \begin{cases} -\dfrac{1}{2}(D_4 z^2 + D_6 y^2) + D(\epsilon)(|y| + |z| + 1) - \dfrac{1}{L} e^{-\alpha|y|}\varphi_3(x) \operatorname{sgn} x, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if} \quad |z + F| \leqslant L, \\ -\tfrac{3}{4}(D_4 z^2 + D_6 y^2) + D(|y| + |z|), \qquad \text{if} \quad |z + F| \geqslant L, \end{cases} \tag{7.4}$$

and, for $|x| \leqslant 1$,

$$V'^* \leqslant \begin{cases} -\tfrac{1}{2}(D_4 z^2 + D_6 y^2) + D(\epsilon)(|y| + |z| + 1), \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if} \quad |z + F| \leqslant L, \\ -\tfrac{3}{4}(D_4 z^2 + D_6 y^2) + D(|y| + |z|), \qquad \text{if} \quad |z + F| \geqslant L. \end{cases} \tag{7.5}$$

It is useful to point out here, with regard to the last term on the first line of the inequality (7.4) for V'^* that the fact that

$$\varphi_3(x) \operatorname{sgn} x \to +\infty \qquad \text{as} \qquad |x| \to \infty$$

does imply the existence of a D such that

$$-\varphi_3(x) \operatorname{sgn} x \leqslant D \qquad \text{for all } x,$$

so that, since $L \geqslant 1$, this term can be majorized thus:

$$-\frac{1}{L} e^{-\alpha|y|}\varphi_3(x) \operatorname{sgn} x \leqslant D, \qquad \text{for all } x, y.$$

Thus, whichever of the inequalities in (7.4) or (7.5) is applicable, it is possible to find a constant D_7 such that

$$V'^* \leqslant -1 \qquad \text{provided that} \qquad y^2 + z^2 \geqslant D_7{}^2. \tag{7.6}$$

It remains now to consider the case when

$$y^2 + z^2 < D_7{}^2. \tag{7.7}$$

Let D_8 be defined by

$$D_8 \equiv D_7 + \max_{|y| \leqslant D_7} |F(y)|,$$

and fix $L = D_8 + 1$ throughout what follows. Assume now that (7.7) holds. Then clearly $|z + F(y)| < L$, and so, provided that $|x| \geqslant 1$, the first inequality in (7.4) is applicable to V'^*, so that if $|x|$ is large enough, say $|x| \geqslant D_9$, to ensure as well that $\varphi_3(x) \operatorname{sgn} x > 0$, we shall then have that

$$V'^* \leqslant -\frac{1}{L} e^{-\alpha D_7} \varphi_3(x) \operatorname{sgn} x + D,$$

for some D. Since $\varphi_3(x) \operatorname{sgn} x \to +\infty$ as $|x| \to \infty$ it is clear from this last estimate that there is a constant $D_{10} \geqslant D_9$ such that

$$V'^* \leqslant -1 \qquad \text{if} \qquad y^2 + z^2 < D_7{}^2 \qquad \text{so long as} \qquad |x| \geqslant D_{10}. \tag{7.8}$$

The results (7.6) and (7.8) show that

$$V'^* \leqslant -1 \qquad \text{if} \qquad x^2 + y^2 + z^2 \geqslant D_7{}^2 + D_{10}^2,$$

which is (3.4) with

$$D_1 \equiv 1 \qquad \text{and} \qquad D_2 \equiv D_7{}^2 + D_{10}^2.$$

This concludes our verification of the theorem, with ϵ and ρ fixed by (7.3).

References

1. R. Reissig, *Math. Nachr.* **32** (1966), 83–88.
2. H. Tejumola, On the boundedness and periodicity of solutions of certain third order differential equations, *Annali di Mat. Pura ed Appl.*, in press.
3. J. O. C. Ezeilo, *Proc. Cambridge Philos. Soc.* **63** (1967), 735–742.